

# Two kinds of quantum adiabatic approximation

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A simple proof of quantum adiabatic theorem is provided. Quantum adiabatic approximation is divided into two kinds. For Hamiltonian  $H(t/T)$ , a relation between the size of the error caused by quantum adiabatic approximation and the parameter  $T$  is given.

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## I. INTRODUCTION

A quantum system is described by its Hamiltonian. When the Hamiltonian is time-independent and its spectral decomposition is known, we can easily get the dynamical evolution operator (DEO). However it is usually impossible to get an analytic expression of the DEO when the Hamiltonian varies with time. In a pioneering paper, Born and Fock considered a kind of time-dependent Hamiltonian  $H_t$  that has a special form  $H_t = H(s)$  where  $s = t/T$  and the spectrum of  $H(s)$  consists of purely discrete eigenvalues [1]. Their result and the extended results [2, 3, 4, 5] on more general  $H(s)$  are all called quantum adiabatic theorem (QAT). For a history of QAT we refer to Ref. [5].

Recently a debate arised on the validity of application of quantum adiabatic theorem (QAT) or quantum adiabatic approximation (QAA) [6, 7, 8, 9, 10, 11, 12]. One thing can be sure is that the widely used simple condition for QAA is actually insufficient [6, 10]. Since QAA and the related Berry phase [13] have a wide application in many fields [14], it is valuable to find new conditions for the approximation. The recent discussion on QAT or QAA is also stimulated by quantum adiabatic computation [15]; quantum computers are believed to be more powerful than computers that we use today [16].

To eliminate ambiguity, the thing should be specified first is the rigorous definitions of QAT and QAA. We only concern time-dependent matrix Hamiltonians without energy degeneracy, which are also the topics of the recent debate. QAT can only be discussed for Hamiltonians that have special form  $H_t = H(s)$  where  $s = t/T$ . This can be seen from the original papers on QAT [1, 4]. We think QAT states that the difference between the DEO and the adiabatic evolution operator (AEO) (defined below) of the system will approach zero in the limit  $T \rightarrow \infty$ . However, we can talk about QAA for Hamiltonians that may not be written in the form  $H_t = H(s)$ . We define QAA as an approximation that uses the AEO in place of the DEO in our calculations. Under this definition of QAA, we find that there are two kinds of QAA and conditions for them to be acceptable are different. Of course we can also discuss QAA for Hamiltonians that are written in the form  $H_t = H(s)$ , one can

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discuss the relation between the amplitude of the error caused by QAA and the value of the parameter  $T$  [17].

The structure of the paper is as follows. In Sec. II, we give an introduction to QAT and present a proof of the theorem. In Sec. III, we give a discussion on QAA. In Sec. IV, we give an example to demonstrate the results obtained in Sec. III. In Sec. V, we give a relation between the size of the error caused by QAA and the parameter  $T$  for Hamiltonian  $H(t/T)$ . Finally the conclusion is given in Sec. VI.

## II. QUANTUM ADIABATIC THEOREM

We just consider non-degenerate two-level systems to demonstrate our basic idea throughout the paper. A smooth time-dependent Hamiltonian  $H_t$  has two instantaneous eigenvalues and eigenstates,

$$H_t |m_t\rangle = m_t |m_t\rangle, \quad m = 1, 2. \quad (1)$$

Eq. (1) cannot determine the phases of the eigenstates  $|m_t\rangle$ . We choose the phases such that [18]

$$\langle m_t | \frac{d}{dt} |m_t\rangle = 0, \quad m = 1, 2. \quad (2)$$

The DEO is denoted by  $U_d(t)$  which satisfies the Schrödinger equation

$$i\hbar \frac{dU_d(t)}{dt} = H_t U_d(t), \quad U_d(0) = I. \quad (3)$$

Except for the trivial cases where the projectors  $|m_t\rangle\langle m_t|$  are independent of time  $t$ , it is usually impossible to obtain an analytic expression for  $U_d(t)$ . The AEO is defined as

$$U_a(t) = \sum_{n=1}^2 e^{-\frac{i}{\hbar} \int_0^t n_{t'} dt'} |n_t\rangle \langle n_0|, \quad (4)$$

which is a little different from the so-called adiabatic transformation [3].

**Theorem 1 (QAT)** *When  $H_t$  has the special form  $H_t = H(s)$  where  $s = t/T \in [0, 1]$ , i.e., it varies from  $H(0)$  to  $H(1)$  using time  $T$ , for any initial state  $|\Psi(0)\rangle$  there is*

$$\lim_{T \rightarrow \infty} \|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| = 0. \quad (5)$$

**Remark.** What we consider is the simplest kind of QAT and it has been proved long ago by considering operator evolutions [1, 2, 3, 4], and no one has doubt on it. From the theorem it is not hard to see that when the system starts from the ground state of  $H(0)$  it will evolve closely to the ground state of  $H(1)$  if  $T$  is big enough; this is the basis of quantum adiabatic computation [15]. Now we give a proof by considering the state vector evolution, which we think is more intuitive, and the process of the proof will be used when we discuss QAA.

**Proof.** We write the system state at time  $t$  as

$$U_d(t) |\Psi(0)\rangle = \sum_{n=1}^2 c_n(t) e^{-\frac{i}{\hbar} \int_0^t n_{t'} dt'} |n_t\rangle. \quad (6)$$

Since  $U_d(t) |\Psi(0)\rangle$  satisfies the Schrödinger equation, we can obtain [18]

$$\frac{dc_1(t)}{dt} = -c_2(t) \langle 1_t | \frac{d}{dt} | 2_t \rangle e^{\frac{i}{\hbar} \int_0^t (1_{t'} - 2_{t'}) dt'}, \quad (7)$$

$$\frac{dc_2(t)}{dt} = -c_1(t) \langle 2_t | \frac{d}{dt} | 1_t \rangle e^{\frac{i}{\hbar} \int_0^t (2_{t'} - 1_{t'}) dt'}. \quad (8)$$

Integrate both sides of Eq. (7) we get

$$\begin{aligned} c_1(t'') - c_1(0) &= - \int_0^{t''} c_2(t) \langle 1_t | \frac{d}{dt} | 2_t \rangle e^{\frac{i}{\hbar} \int_0^t (1_{t'} - 2_{t'}) dt'} dt \\ &= - \int_0^{t''} c_2(t) \frac{\langle 1_t | \frac{d}{dt} | 2_t \rangle}{i(1_t - 2_t)/\hbar} d \left[ e^{\frac{i}{\hbar} \int_0^t (1_{t'} - 2_{t'}) dt'} \right]. \end{aligned} \quad (9)$$

In the above we have used the fact that there is no energy degeneracy, i.e.,  $1_t - 2_t \neq 0$ . Integrate by part we get

$$c_1(t'') - c_1(0) = A(t'') + B(t'') + C(t''), \quad (10)$$

where

$$A(t'') = -c_2(t) f_t e^{\frac{i}{\hbar} \int_0^t (1_{t'} - 2_{t'}) dt'} \Big|_{t=0}^{t=t''}, \quad (11)$$

$$B(t'') = \int_0^{t''} c_2(t) \frac{df_t}{dt} e^{\frac{i}{\hbar} \int_0^t (1_{t'} - 2_{t'}) dt'} dt, \quad (12)$$

$$C(t'') = \int_0^{t''} \left[ \frac{dc_2(t)}{dt} f_t e^{\frac{i}{\hbar} \int_0^t (1_{t'} - 2_{t'}) dt'} \right] dt, \quad (13)$$

$$f_t = \frac{\langle 1_t | \frac{d}{dt} | 2_t \rangle}{i(1_t - 2_t)/\hbar}. \quad (14)$$

Because  $H(s)$  depends only on the parameter  $s$ , instantaneous eigenstates and eigenvalues are also dependent only on  $s$ , i.e., we can write  $|m_t\rangle = |m(s)\rangle$ ,  $m_t = m(s)$ . Now we have

$$f_t = \frac{1}{T} \frac{\hbar \langle 1(s) | \frac{d}{ds} | 2(s) \rangle}{i(1(s) - 2(s))} = \frac{f(s)}{T}. \quad (15)$$

From Eqs. (11-13) we can get

$$|A(t'')| \leq \frac{2}{T} \max_{s \in [0,1]} |f(s)|, \quad (16)$$

$$|B(t'')| \leq \frac{1}{T} \max_{s \in [0,1]} \left| \frac{d}{ds} f(s) \right|, \quad (17)$$

$$|C(t'')| \leq \frac{1}{T} \max_{s \in [0,1]} \left| \langle 2(s) | \frac{d}{ds} | 1(s) \rangle f(s) \right|. \quad (18)$$

In deriving Eq. (18) from Eq. (13) we have used Eq. (8). From Eq. (10), we know  $|A(t'')| + |B(t'')| + |C(t'')|$  is an upper bound of  $|c_1(t'') - c_1(0)|$ , and Eqs. (16-18) indicate this upper bound will approach zero in the limit  $T \rightarrow \infty$ , so we have

$$\lim_{T \rightarrow \infty} |c_1(t) - c_1(0)| = 0. \quad (19)$$

Similarly we can obtain

$$\lim_{T \rightarrow \infty} |c_2(t) - c_2(0)| = 0. \quad (20)$$

Because

$$\begin{aligned} & \|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| \\ &= \sqrt{\sum_{n=1}^2 |c_n(t) - c_n(0)|^2}, \end{aligned} \quad (21)$$

we get

$$\lim_{T \rightarrow \infty} \|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| = 0, \quad (22)$$

which completes the proof. ■

### III. QUANTUM ADIABATIC APPROXIMATION

The system state  $|\Psi(t)\rangle$  at time  $t$  and the initial state  $|\Psi(0)\rangle$  are connected by the relation  $|\Psi(t)\rangle = U_d(t) |\Psi(0)\rangle$ . Suppose the operator  $B(t)$  represents a physical quantity at time  $t$ . Quantum mechanics tells us that when we measure the quantity the result will be random and the average value will be

$$\langle \Psi(t) | B(t) | \Psi(t) \rangle = \langle \Psi(0) | U_d^\dagger(t) B(t) U_d(t) | \Psi(0) \rangle. \quad (23)$$

Assume we are given the initial state  $|\Psi(0)\rangle$  and the Hamiltonian  $H_t$  of the system. If we cannot figure out  $U_d(t)$  from  $H_t$ , usually we will not know the average value  $\langle \Psi(t) | B(t) | \Psi(t) \rangle$ . From QAT we know that in some cases the difference between  $U_d(t)$  and  $U_a(t)$  will be very small, we may consider whether it is acceptable to use  $U_a(t)$  in place of  $U_d(t)$  in calculating the average in Eq. (23). We define QAA as an approximation that uses the AEO  $U_a(t)$  in place of the DEO  $U_d(t)$  in our calculations. The definition of QAA applies to a general time-dependent  $H_t$  contrast to QAT. If we require the approximation to be acceptable for any physical quantity  $B(t)$  and any initial state  $|\Psi(0)\rangle$  in Eq. (23), it means the difference between  $U_d(t) |\Psi(0)\rangle$  and  $U_a(t) |\Psi(0)\rangle$  should be small.

The difference between  $U_d(t) |\Psi(0)\rangle$  and  $U_a(t) |\Psi(0)\rangle$  is small means

$$\|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| \ll 1, \quad (24)$$

which is equivalent to

$$\langle \Psi(0) | U_a^\dagger(t) U_d(t) | \Psi(0) \rangle \approx 1. \quad (25)$$

Due to the linearity of quantum mechanics, for arbitrary initial state  $|\Psi(0)\rangle$ , condition (25) will be satisfied when

$$\langle m_0 | U_a^\dagger(t) U_d(t) | m_0 \rangle \approx 1, \quad m = 1, 2, \quad (26)$$

which we regard as the condition for the first kind of QAA. This kind of QAA pays attention to the phase of  $U_d(t)|m_0\rangle$ . Condition (26) ensures that the relative phase between the two instantaneous eigenstates in  $U_d(t)|\Psi(0)\rangle$  is almost the same as that in  $U_a(t)|\Psi(0)\rangle$ .

The probability of finding the system in an instantaneous energy eigenstate  $|n_t\rangle$  is of considerable interest, e.g., in coherent population transfer among quantum states of atoms and molecules [19]. In this case, in Eq. (23) the operator  $B(t) = |n_t\rangle\langle n_t|$  and the probability is  $|\langle n_t|U_d(t)|\Psi(0)\rangle|^2$ . From the definition of the  $U_a(t)$  in Eq. (4) we know  $|\langle n_t|U_a(t)|\Psi(0)\rangle|^2$  is a constant, it means the system will follow the instantaneous eigenstate if it starts from an instantaneous eigenstates. Therefore it is acceptable to use  $U_a(t)$  in place of  $U_d(t)$  in calculating the probability  $|\langle n_t|U_d(t)|m_0\rangle|^2$  when

$$|\langle m_t|U_d(t)|m_0\rangle| \approx 1, \quad m = 1, 2. \quad (27)$$

Eq. (27) is the same as

$$|\langle m_0|U_a^\dagger(t)U_d(t)|m_0\rangle| \approx 1, \quad m = 1, 2, \quad (28)$$

which we regard as the condition for the second kind of QAA. This kind of QAA is considered in [4, 18]. When condition (28) is satisfied, the relative phase between the two instantaneous eigenstates in  $U_d(t)|\Psi(0)\rangle$  may differ much from the relative phase in  $U_a(t)|\Psi(0)\rangle$ .

The second kind of QAA is just a special case of the first kind and the conditions for them are different: condition (26) can lead to (28) while (28) may not lead to (26), so there are cases where the second kind of QAA is acceptable while the first kind of QAA is unacceptable. Though the conditions for two kinds of QAA are given, usually it is not easy to directly check whether they are satisfied. In the following we will give a discussion on the conditions for them.

From (21) we know that if

$$|c_n(t) - c_n(0)| \ll 1, \quad n = 1, 2, \quad (29)$$

the first kind of QAA will be acceptable. From Eq. (10) we can get

$$|c_1(t'') - c_1(0)| \leq |\bar{A}(t'')| + |\bar{B}(t'')| + |\bar{C}(t'')|, \quad (30)$$

where

$$|\bar{A}(t'')| = \left| \frac{\hbar \langle 1_t | \frac{d}{dt} | 2_t \rangle}{1_t - 2_t} \right|_{t=0} + \left| \frac{\hbar \langle 1_t | \frac{d}{dt} | 2_t \rangle}{1_t - 2_t} \right|_{t=t''}, \quad (31)$$

$$|\bar{B}(t'')| = t'' \max_{t \in [0, t'']} \left| \frac{d}{dt} \left( \frac{\hbar \langle 1_t | \frac{d}{dt} | 2_t \rangle}{1_t - 2_t} \right) \right|, \quad (32)$$

$$|\bar{C}(t'')| = t'' \max_{t \in [0, t'']} \left| \langle 2_t | \frac{d}{dt} | 1_t \rangle \frac{\hbar \langle 1_t | \frac{d}{dt} | 2_t \rangle}{1_t - 2_t} \right|. \quad (33)$$

Eq. (30) gives an upper bound for  $|c_1(t'') - c_1(0)|$ . The first kind of QAA will be acceptable if  $|\bar{A}(t'')|$ ,  $|\bar{B}(t'')|$  and  $|\bar{C}(t'')|$  are all very small. Eqs. (32) and (33) indicate that it may be inappropriate to use  $U_a(t)|\Psi(0)\rangle$  in place of  $U_d(t)|\Psi(0)\rangle$  when  $t$  is large. When instantaneous eigenstates and eigenvalues are given, it may be easy to check whether  $|\bar{A}(t'')|$ ,  $|\bar{B}(t'')|$  and  $|\bar{C}(t'')|$  are small at the given time  $t''$ . For the example we will discuss in the next section, this method is quite good.

The second kind of QAA is discussed in Refs. [4, 18]. It is known that when

$$\left| \int_0^t \langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle e^{\frac{i}{\hbar} \int_0^{t_1} (m_{t'} - n_{t'}) dt'} dt_1 \right| \ll 1, \quad n \neq m, \quad (34)$$

the second kind of QAA will be acceptable [4, 18]. We emphasize that the instantaneous eigenstates in (34) satisfy the phase condition (2). We can derive Eq. (34) as follows. It is not hard to know that the unitary operator  $\tilde{U}(t) = U_a^\dagger(t) U_d(t)$  is generated by the Hamiltonian

$$\tilde{H}_t = -i\hbar \sum_{m \neq n}^M \langle m_t | \frac{d}{dt} | n_t \rangle e^{\frac{i}{\hbar} \int_0^t (m_{t'} - n_{t'}) dt'} | m_0 \rangle \langle n_0 |. \quad (35)$$

We can get an expansion of  $\tilde{U}(t)$ ,

$$\tilde{U}(t) = I + \frac{1}{i\hbar} \int_0^t dt_1 \tilde{H}_{t_1} + \frac{1}{(i\hbar)^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{H}_{t_1} \tilde{H}_{t_2} + \dots \quad (36)$$

Substitute Eq. (4) and Eq. (36) into the expression  $U_d(t) = U_a(t) \tilde{U}(t)$ , we can obtain an expansion of  $U_d(t)$ . Similar discussions appear in Refs. [20, 21]. The condition (28) of the second kind of QAA is equivalent to

$$\left| \langle m_0 | U_a^\dagger(t) U_d(t) | n_0 \rangle \right| = \left| \langle m_0 | \tilde{U}(t) | n_0 \rangle \right| \ll 1, \quad m \neq n. \quad (37)$$

When we just substitute the first two terms of Eq. (36) into (37), we will obtain the condition (34). Though (34) as a condition for the second kind of QAA has not been strictly proved since we just use the first two terms in Eq. (36), we believe it is sufficient. If  $\langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle$  and  $(m_{t'} - n_{t'})$  are constants, condition (34) can be simplified into

$$\left| \frac{\hbar \langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle}{(m_{t_1} - n_{t_1})} \right| \ll 1, \quad m \neq n, \quad (38)$$

while when  $\langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle$  and  $(m_{t'} - n_{t'})$  are not constants, a modification of (38) such as

$$\max_{t_1 \in [0, t]} \left| \frac{\hbar \langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle}{m_{t_1} - n_{t_1}} \right| \ll 1, \quad m \neq n, \quad (39)$$

cannot replace (34) as a sufficient condition for the second of QAA [6, 10, 18, 22], however it may be acceptable to regard (39) as a sufficient condition when  $\langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle$  and  $(m_{t'} - n_{t'})$  vary slowly. The counterexample in Ref. [6] demonstrates that a contradiction will appear at a special evolution time if we take (39) as a sufficient condition. We think the rapid changing of  $\langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle$  causes the contradiction because when  $\langle m_{t_1} | \frac{d}{dt_1} | n_{t_1} \rangle$  varies fast, especially when its varying frequency resonant to the energy gap, condition (39) deviates much from (34). Finally we want to emphasize that even when condition (34) is satisfied, it doesnot means it is acceptable to use  $U_a(t) | n_0 \rangle$ , condition to approximate  $U_d(t) | n_0 \rangle$ .

#### IV. AN EXAMPLE

In this section we will give an example to demonstrate that there are cases where the second kind of QAA is acceptable while the first kind is unacceptable.

Consider the Hamiltonian

$$H_t = -\hbar\omega_0 [\sigma_x \cos 2\omega t + \sigma_y \sin 2\omega t], \quad (40)$$

where  $\omega_0$  and  $\omega$  are both positive. Its eigenvalues are  $1_t = \hbar\omega_0$  and  $2_t = -\hbar\omega_0$ . We choose the eigenstates

$$\begin{aligned} |1_t\rangle &= (e^{-i\omega t} |0\rangle - e^{i\omega t} |1\rangle) / \sqrt{2}, \\ |2_t\rangle &= (e^{-i\omega t} |0\rangle + e^{i\omega t} |1\rangle) / \sqrt{2}, \end{aligned} \quad (41)$$

which satisfy the phase condition (2). It can be proved that in the basis  $\{|1_0\rangle, |2_0\rangle\}$

$$U_a^\dagger(t) U_d(t) = \begin{bmatrix} e^{i\omega_0 t} \left( \cos \bar{\omega} t - \frac{i\omega_0}{\bar{\omega}} \sin \bar{\omega} t \right) & e^{i\omega_0 t} \times \frac{i\omega}{\bar{\omega}} \sin \bar{\omega} t \\ e^{-i\omega_0 t} \times \frac{i\omega}{\bar{\omega}} \sin \bar{\omega} t & e^{-i\omega_0 t} \left( \cos \bar{\omega} t + \frac{i\omega_0}{\bar{\omega}} \sin \bar{\omega} t \right) \end{bmatrix}, \quad (42)$$

where  $\bar{\omega} = \sqrt{\omega^2 + \omega_0^2}$ .

The condition (28) for the second kind of QAA can be written as

$$(\cos \bar{\omega} t)^2 + \left( \frac{\omega_0}{\bar{\omega}} \sin \bar{\omega} t \right)^2 \approx 1, \quad (43)$$

which can be simplified into

$$\left( \left( \frac{\omega_0}{\bar{\omega}} \right)^2 - 1 \right) \sin^2 \bar{\omega} t \approx 0. \quad (44)$$

From (44) it can be seen that when  $\omega \ll \omega_0$  the second kind of QAA will be acceptable no matter how large  $t$  is. This result can also be obtained from Eq. (34).

The condition (26) for the first kind of QAA can be written as

$$e^{i\omega_0 t} \left( \cos \bar{\omega} t - \frac{i\omega_0}{\bar{\omega}} \sin \bar{\omega} t \right) \approx 1, \quad (45)$$

which is the same as

$$e^{-i\frac{\omega^2 t}{\bar{\omega} + \omega_0}} + i e^{i\omega_0 t} \left( 1 - \frac{\omega_0}{\bar{\omega}} \right) \sin \bar{\omega} t \approx 1. \quad (46)$$

Notice that (45) can lead to (43), it is not hard to see that (46) is equivalent to (44) plus  $\omega^2 t / (\bar{\omega} + \omega_0) \ll 1$ . When  $\omega \ll \omega_0$  and  $\omega^2 t / \omega_0 \ll 1$  the first kind of QAA will be acceptable. Though condition  $\omega \ll \omega_0$  is enough for the second kind of QAA to be acceptable, it cannot certainly make the first kind of QAA acceptable when  $t$  is large and this result has been implied in Ref. [23].

In this example, the upper bound expression Eq. (30) can be written as

$$|c_1(t) - c_1(0)| \leq \frac{\omega}{\omega_0} + \frac{\omega^2 t}{2\omega_0}. \quad (47)$$

From (47) we can also conclude that when  $\omega \ll \omega_0$  and  $\omega^2 t / \omega_0 \ll 1$  the first kind of QAA will be acceptable, which coincides with the result obtained by figuring out  $U_a^\dagger(t) U_d(t)$ .

## V. QUANTUM ADIABATIC APPROXIMATION FOR HAMILTONIAN $H(t/T)$

The above discussion on QAA applies to a general time-dependent  $H_t$ . When  $H_t$  has the special form  $H_t = H(s)$  where  $s = t/T \in [0, 1]$ , we can discuss the relation between the parameter  $T$  and the error caused by QAA.

**Theorem 2** Suppose  $H_t$  has the special form  $H_t = H(s)$  where  $s = t/T \in [0, 1]$ . When

$$T \geq \frac{\sqrt{2}\hbar}{\delta} \max_{s \in [0,1]} \left( \frac{2 \left\| \frac{dH(s)}{ds} \right\|}{\Delta(s)^2} + \frac{7 \left\| \frac{dH(s)}{ds} \right\|^2}{\Delta(s)^3} + \frac{\left\| \frac{d^2 H(s)}{ds^2} \right\|}{\Delta(s)^2} \right) \quad (48)$$

where  $\Delta(s) = |1(s) - 2(s)|$ , the error caused by QAA cannot be bigger than  $\delta$ , i.e.,

$$\|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| \leq \delta. \quad (49)$$

**Proof.** In Sec. II, we write  $U_d(t) |\Psi(0)\rangle$  in Eq. (6) and we know from Eqs. (10,16-18) that

$$\begin{aligned} & |c_n(t) - c_n(0)| \\ & \leq \frac{1}{T} \max_{s \in [0,1]} \left[ 2|f(s)| + \left| \frac{df(s)}{ds} \right| + \left| \langle 2(s) | \frac{d}{ds} |1(s)\rangle f(s) \right| \right], \end{aligned} \quad (50)$$

where

$$f(s) = \frac{\hbar \langle 1(s) | \frac{d}{ds} |2(s)\rangle}{i(1(s) - 2(s))} \quad (51)$$

is defined in Eq. (15). Now we give an upper bound of  $|c_n(t) - c_n(0)|$  expressed in norm of the Hamiltonian  $H(s)$  and its derivatives. First we have [4, 18]

$$\langle 1(s) | \frac{d}{ds} |2(s)\rangle = \frac{\langle 1(s) | \frac{dH(s)}{ds} |2(s)\rangle}{2(s) - 1(s)}, \quad (52)$$

so there are

$$|f(s)| = \left| \frac{\hbar \langle 1(s) | \frac{dH(s)}{ds} |2(s)\rangle}{-i(1(s) - 2(s))^2} \right| \leq \frac{\hbar \left\| \frac{dH(s)}{ds} \right\|}{\Delta(s)^2}, \quad (53)$$

and

$$\left| \langle 2(s) | \frac{d}{ds} |1(s)\rangle f(s) \right| \leq \frac{\hbar \left\| \frac{dH(s)}{ds} \right\|^2}{\Delta(s)^3}. \quad (54)$$

It can be proved that

$$\begin{aligned} \left| \frac{d}{ds} f(s) \right| &= \left| \frac{d}{ds} \left[ \frac{\hbar \langle 1(s) | \frac{dH(s)}{ds} |2(s)\rangle}{-i(1(s) - 2(s))^2} \right] \right| \\ &\leq \frac{\hbar \left\| \frac{d^2 H(s)}{ds^2} \right\|}{\Delta(s)^2} + 6 \frac{\hbar \left\| \frac{dH(s)}{ds} \right\|^2}{\Delta(s)^3}. \end{aligned} \quad (55)$$



Substitute Eqs. (53-55) into Eq. (50), we get

$$|c_n(t) - c_n(0)| \leq \frac{\hbar}{T} \max_{s \in [0,1]} \left[ \frac{2 \left\| \frac{dH(s)}{ds} \right\|}{\Delta(s)^2} + \frac{\left\| \frac{d^2 H(s)}{ds^2} \right\|}{\Delta(s)^2} + \frac{7 \left\| \frac{dH(s)}{ds} \right\|^2}{\Delta(s)^3} \right]. \quad (56)$$

Because

$$\begin{aligned} & \|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| \\ &= \sqrt{\sum_{n=1}^2 |c_n(t) - c_n(0)|^2}, \end{aligned} \quad (57)$$

when (48) is satisfied we have

$$\begin{aligned} & \|U_d(t) |\Psi(0)\rangle - U_a(t) |\Psi(0)\rangle\| \\ &\leq \frac{\hbar\sqrt{2}}{T} \max_{s \in [0,1]} \left[ \frac{2 \left\| \frac{dH(s)}{ds} \right\|}{\Delta(s)^2} + \frac{\left\| \frac{d^2 H(s)}{ds^2} \right\|}{\Delta(s)^2} + \frac{7 \left\| \frac{dH(s)}{ds} \right\|^2}{\Delta(s)^3} \right] \\ &\leq \delta, \end{aligned} \quad (58)$$

which completes the proof.

A result similar to theorem 2 appears in Ref. [17], which is derived from a very different method. Under the same error  $\delta$ , the required evolution time  $T$  in our result is proportional to  $1/\delta$  while in Ref. [17] it is proportional to  $1/\delta^2$ . We think this is one of the most differences between the two results. ■

## VI. CONCLUSION

We give a discussion on the conditions of QAA. We present a proof of QAT, which we think is easier than that appears in the textbook [4]. We think there are two kinds of QAA, one cares the relative phase in the approximate system state while the other does not. For the kind of Hamiltonian  $H(t/T)$  we give a relation between the size of the error caused by QAA and the parameter  $T$ .

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